Supermodularity and Search

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Abstract

This paper studies worker-firm sorting under search frictions. I show that complementarities in the match production function have robust and observable implications, in the form of a single-crossing property in hiring rates. The single-crossing property, while weaker than positive assortative matching, still captures the intuition that better workers prefer to work for better firms and worse workers with worse firms. The single-crossing result holds in the canonical random search model of Shimer and Smith (2000), as well as models of directed search. The single-crossing property also holds for an entire family "partially directed" search models, where workers allocate their search effort across many firm types. The single-crossing result easily generalizes to models of multi-worker firms with decreasing returns to scale production. The robustness of this result is in contrast with the existing literature on sufficient conditions for positive assortative matching, which tend to be both more demanding and depend on the details of the search technology.

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1 Introduction

Under what conditions do more productive workers match with more productive firms? What is the connection between the production function and observed worker-firm matching patterns? To what extent do search frictions change matching patterns? These questions have received significant attention in recent years. Many papers have studied extensions of the model in Becker (1973), adding search frictions and finding conditions that guerentee positive assortative matching. This paper identifies a new property of worker-firm matching patterns, *weak single crossing in matching*, and shows that it is tightly related to the match production function. Specifically, if the match production function is supermodular (i.e. there are positive complementaries) then we will observe weak single crossing in matching, which intuitively means that high type firms tend to match with higher type workers more than low type firms do. I show that this connection holds in the model of Shimer and Smith (2000), as well as models of directed search, and in a flexible class of search models where workers allocate differential search effort to each firm type. This last class models is intended to capture the empirically relevant case where worker search is neither purely random nor fully directed to single firm types. The last sections of the paper generalize the weak single crossing result further, to models of search with multi-worker worker firms with decreasing returns to scale, similar to the workhorse models used in international trade and macroeconomics.

The main result connects supermodularity of the production function to weak single crossing in matching. Supermodularity simply means that there are complementarities in production between worker types and firm types: The gain in output from substituting a better worker for a worse worker is larger at higher-type firms. Weak single crossing in matching requires more explanation. Let $y < y'$ be two firm types, with y' being the more productive firm. Let $m(x, y)$ be the matching rate of type x workers with type y firms. Again, high x workers are more productive. (Strong) single crossing in matching rates means that $m(\cdot, y')$ cuts $m(\cdot, y)$ once and from below. This means that low type unemployed workers transition to type y' firms at a lower rate than as compared with type y firms.

Above the crossing point, workers match with y' more often. This naturally captures the idea that high type workers have a greater affinity for type y' firms, and low type workers have a greater affinity for type y firms. *Weak* single crossing in matching rates is a slightly weaker condition that captures the same idea. Weak single crossing in matching rates means that $m(\cdot, y')$ never cuts $m(\cdot, y)$ from above. Thus $m(\cdot, y')$ may equal $m(\cdot, y)$ at multiple points or multiple intervals, but once $m(x, y')>m(x, y)$, we never obvserve $m(x', y') < m(x', y)$ for $x' > x$. As discussed below, weak single crossing in matching is related to, but weaker than, positive assortative matching as defined by Shimer and Smith (2000).

In his seminal contribution Becker (1973) pointed out that supermodular production is sufficient for positive assortative matching in a frictionless environment. Weak single crossing in matching generalizes this result, giving the implications of supermodular production in a variety of frictional environments. Thus the approach here is different from much of the literature, which has focused on finding conditions sufficient to generate positive assortative matching, rather than characterizing the observable implications of simple supermodularity (see, e.g., Shimer and Smith (2000), Shimer (2005), Shi (2001), Lentz (2010), Eeckhout and Kircher (2010), Peters (2010), Eeckhout and Kircher (2012)).

The baseline model in Section 2 is a generalized model of directed search. Workers can allocate search to many different types of firms. There are decreasing returns to search for each firm type, so workers optimally spread their effort across firm types, allocating more effort to firms that offer more match surplus. Thus, worker search is neither fully random nor fully directed. This seems to be the empirically relevant case, as anecdotally workers do not fully focus their search efforts on a single type of potential employer, nor do they simply sample opportunities randomly. I show that supermodular production implies weak single crossing in matching, even in this relatively rich environment. Several other papers (Restrepo-Echavarria et al. (2013), Menzio (2007), Lester (2011)) have modeled intermediate cases of search frictions, between random search and directed search. This paper is the first to derive analytical sorting results in such an environment.

Of course, some structure is needed to derive these results. In the baseline model, I rely on linear vacancy posting costs, and firms fully directing their recruitment efforts. This assumption means that vacancy posting in a particular submarket is a function of the arrival rate and surplus in that market only. Recruiting activity in other markets does not directly affect the choice of vacancies. These assumptions are strong, but are fairly standard in the literature. The assumption of linear vacancy posting costs, in particular, is the default, and can be justified as reflecting the ability of firms to buy recruitment services in a competitive market.

Having established the basic result, in Section 3 I show that the model includes several existing models as special cases, so the single crossing results extends to those model as well. I also show that the results extend to the random search model of Shimer and Smith (2000) (which does not have linear vacancy posting costs). While the baseline model is of single worker firms, one of the main contributions of this paper is to show, in Section 4, that the results generalize to multi-worker firms. It is known that when firms employ a single worker, or when production is linear with a penalty for the number of employees, supermodularity is sufficient for positive assortative matching. Much of the frictional sorting literature has been developed by adding search frictions to this type of model. In contrast, when production has decreasing returns in an aggregate of worker output (Costinot (2009), Costinot and Vogel (2010)) then log-supermodularity is the sufficient condition. This class of models is standard in trade and macroeconomics. In Section 4 I show that the sufficient conditions for positive assortative matching in frictionless multi-worker models are sufficient for weak single crossing in matching under search frictions. If production is based on Becker (1973), or multi-worker versions of Becker (1973), then supermodularity implies weak single crossing in matching under search frictions. If production is based on Costinot (2009) then log-supermodularity implies weak single crossing in matching under search frictions.

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2 Baseline Model

Time is continuous and goes on forever. The economy is populated by a fixed mass of workers. Worker types are indexed by $x \in [0,1]$ and firm types are indexed by $y \in [0,1]$, with higher type agents being productive. Let $l(x)$ be the mass of type x workers, with the total mass of workers being $L = \int_0^1 l(x)dx$.

Workers may be employed or unemployed. Jobs may be vacant or filled. Firms operate constant returns production technologies, and create jobs freely subject to a vacancy posting cost c. Only unemployed workers and vacant jobs can search for new partners. Let $u(x)$ and $v(y)$ be the mass of unemployed type x workers and the mass of type y vacant jobs, respectively. Workers and firms search for partners in frictional *submarkets*, which are specific to the types of agents searching in them. Thus a submarket can be identified by the tuple (x, y). A worker of type x' can only search in markets with worker type x', and firms are similarly restricted. Let $v(x, y)$ be the mass of vacancies opened in the (x, y) submarket. On the worker side, each worker chooses whether to search in a given submarket and how much effort to expend in the submarket. Let $u(x, y)$ be the mass of workers participating in submarket (x, y), and let $e(x, y)$ be the per-worker effort in that submarket. Conditional on participating in a given submarket, effort is increasingly costly on the margin. This convexity in costs incentivizes workers to spread their effort across many markets, in contrast to the textbook directed search models, where search is focused on a single submarket.

When matched, a worker and firm produce output $f(x, y)$ per unit of time. The results derived below do not require $f(x, y)$ to be increasing in both arguments (i.e. absolute advantage), but it is often easiest to consider that case, since it make clear what being a better worker or firm means. The surplus of the match is split according to generalized Nash bargaining, with the worker receiving the exogenous fraction η of match surplus. Matches dissolve at the exogenous rate s, and there is no endogenous match destruction.

Let $U(x)$ be the value function of an unemployed type x worker, and $W(x, y)$ the value of a worker matched with a type y firm. The value function of an unemployed worker is then

$$rU(x) = -\int k(e(x,y'))dy' + \int e(x,y')p(x,y')\left(W(x,y') - U(x,y')\right)dy' \tag{1}$$

Here $k(\cdot)$ is a positive, increasing and convex function capturing the increasing marginal cost of expending effort in each submarket. The convexity of $k(\cdot)$ means that it is increasing difficult for the worker to elicit meetings with a given firm type. $p(x,y)$ is the arrival rate of offers, per unit of effort, in the (x,y) submarket. $p(x,y)$ is exogenous to an individual worker, but is the outcome of the equilibrium behavior of type x workers and type y firms. The first term on the left hand side of equation (1) is the flow cost to the worker of searching: the sum of costs across all submarkets the worker searches in. The second term is the expected benefit benefit of searching: the value of jobs, $W(x,y') - U(x,y')$, weighted by their arrival rates. The worker sets $e(x,y)$ in each submarket so that the marginal cost equals the marginal benefit:

$$k'(e(x,y)) = p(x,y)\left(W(x,y) - U(x,y)\right) \qquad if \quad e(x,y) > 0. \tag{2}$$

This will be critical later.

The value of an employed worker is simpler, simply the value of wages less the loss in utility coming from exogenous job destruction:

$$rW(x,y) = w(x,y) - s\left(W(x,y) - U(x,y)\right) \tag{3}$$

On the firm side, let $V(x,y)$ be the value of a type y vacancy in the (x,y) submarket. Let $J(x,y)$ be the value to the firm of a type y job filled by a type x worker. Vacancies are posted at cost c. The value of a vacancy is given by

$$rV(x,y) = -\gamma + q(x,y)\left(J(x,y) - V(x,y)\right)$$

where $q(x,y)$ is the arrival rate of workers a vacancies. Unlike unemployed workers, vacancies can enter freely subject to the linear posting cost γ. Thus in equilibrium $V(x,y) = 0$ in any active submarket, implying

$$\frac{\gamma}{q(x,y)} = J(x,y) \qquad if \quad v(x,y) > 0. \tag{4}$$

The value of a filled job is then

$$rJ(x,y) = f(x,y) - w(x,y) - sJ(x,y) \tag{5}$$

where $f(x,y) - w(x,y)$ is the flow match output less wages, and $s\,(J(x,y) - V(x,y))$ is the loss due to exogenous match destruction.

Let

$$S(x,y) = W(x,y) + J(x,y) - U(x)$$

be total match surplus. Combining equations (1), (3) and (5) I have

$$S(x,y) = \frac{1}{r+s}\left(f(x,y) - rU(x)\right) \tag{6}$$

This equation describes match surplus as the difference between flow production and the value of unemployment. The important thing here is that the expression is additive. This implies that the cross partial $S_{12}(x,y) = \frac{1}{r+s} f_{12}(x,y)$ has the same sign as $f_{12}(x,y)$, so that complementarities in the production function carry through to complementarities in the surplus function.

Next, consider the bargaining process. By Nash bargaining, each agent gets a share of total match surplus:

$$W(x,y) - U(x,y) = \eta S(x,y)$$

$$J(x,y) = (1-\eta)S(x,y)$$

Substituting these expressions into (2) and (4), I have

$$k'(e(x,y)) = p(x,y)\eta S(x,y) \qquad if \quad e(x,y) > 0 \tag{7}$$

$$\frac{\gamma}{q(x,y)} = (1-\eta)S(x,y) \qquad if \quad v(x,y) > 0. \tag{8}$$

These two equations describe the relationships between surplus, arrival rates, and search effort. To make further progress I must be explicit about the matching function which determines $p(x,y)$ and $q(x,y)$. I make the standard assumptions, that the flow of matches in submarket (x,y) is given by

$$m\left(u(x)e(x,y), v(x,y)\right),$$

an increasing, concave and linearly homogeneous function. In addition, $m\left(0, v(x,y)\right) = m\left(u(x)e(x,y), 0\right) = 0$. The first argument to m is total worker search effort: effort per worker $e(x,y)$ times the number of searchers $u(x)$. The second is the total mass of vacancies in the submarket. The arrival rates $p(x,y)$ and $q(x,y)$ are just the total flow of matches divided by the appropriate measure of search effort:

$$p(x,y) = \frac{m\left(u(x)e(x,y), v(x,y)\right)}{u(x)e(x,y)}$$

$$q(x,y) = \frac{m\left(u(x)e(x,y), v(x,y)\right)}{v(x,y)}$$

By constant returns, I can rewrite (7) and (8) as

$$k'(e(x,y)) = m\left(1, \theta(x,y)\right)\eta S(x,y) \qquad if \quad e(x,y) > 0 \tag{9}$$

$$\gamma = \frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)}(1-\eta)S(x,y) \qquad if \quad v(x,y) > 0 \tag{10}$$

In can be shown (in the appendix) that all positive surplus submarkets will have $e(x,y) > 0$ and $v(x,y) > 0$, so (9) and (10) can be written as

$$k'(e(x,y)) = m\,(1,\theta(x,y))\,\eta S(x,y) \qquad if \quad e(x,y) > 0 \tag{11}$$

$$\gamma = \frac{m\,(1,\theta(x,y))}{\theta(x,y)}(1-\eta)S(x,y) \qquad if \quad v(x,y) > 0 \tag{12}$$

2.0.1 Off-equilibrium beliefs

Equations (9) and (10) specify $v(x,y)$ and $e(x,y)$ conditional the sumarket being operational, i.e., $\theta(x,y) > 0$. It is clear that $v(x,y) = e(x,y) = 0$ whenever $S(x,y) \leq 0$, since there would be no surplus to cover the costs of search. The question asrises of whether all submarkets with $S(x,y) > 0$ will be operational. It turns out they will, under a standard assumption concerning off-equilibrium beliefs. Note that (9) and (10) do not say anything about markets where $v(x,y) = e(x,y) = 0$. In other words, they say nothing about what a firm would expect to happen if they played an off-equilibrium strategy and posted vacancies to a closed market, so I must make an assumption about the off-equilibrium beliefs of the firms. Let (x,y) be a submarket with no workers searching and no vacancies in equilibrium. The question is what will happen if a firm of type y deviates and posts vacancies in (x,y). The outcome would be that workers respond by expending effort in (x,y) until (9) is satisfied. That is, the firm believes that workers will maximize their utility conditional on the firm's vacancy choice. This is a standard assumption in directed search models.

Lemma 1. *Assume that the firm, when contemplating a deviation in their vacancy posting choice, believes that workers will maximize their utility conditional on the firm's vacancy choice. Then $S(x,y)$ implies $v(x,y) > 0$ and $e(x,y) > 0$.*

Proof of Lemma 1. Note that by assumption $k'(0) = 0$. This means that $e(x,y) > 0$ if and only if $S(x,y) > 0$, and $e(x,y) = 0$ otherwise. So I can write

$$k'(e(x,y)) = m\,(1,\theta(x,y))\,\eta S(x,y) \qquad if \quad S(x,y) > 0. \tag{13}$$

If $S(x,y) > 0$ then workers can be expected to meet any strictly positive value of of $v(x,y)$ with strictly positive $e(x,y)$. Now assume that there is a submaket (x,y) with $S(x,y) > 0$ and $v(x,y) = e(x,y) = 0$. Then the first order condition (9) holds:

$$0 = k'(e(x,y)) - m\left(1, \frac{v(x,y)}{u(x)e(x,y)}\right)\eta S(x,y) \tag{14}$$

Equation (14) implicitly defines $e(x,y)$ as a function of $v(x,y)$, holding $S(x,y)$ and $u(x)$ constant. Implicitly differentiating (14) I get the elasticity as

$$\frac{de}{dv}\frac{v}{e} = \frac{m_2\left(1, \frac{v}{ue}\right)\frac{v}{ue}\eta S(x,y)}{k''(e)e + m_2\left(1, \frac{v}{ue}\right)\frac{v}{ue}\eta S(x,y)} < 1$$

where the inequality follows from This means that a certain percent change in v leads to a *smaller* percent change in e. Thus as v shrinks, e shrinks proportionally less, and $\frac{v}{ue} = \theta \to 0$. What this means is that we can always find a sufficiently small v such that

$$\gamma = \frac{m(1, \theta(x,y))}{\theta(x,y)}(1-\eta)S(x,y) \tag{15}$$

since we can always drive $\theta \to 0$, making worker arrivals arbitrarly high, so that it is worth it to post in that market. Thus, we cannot have $S(x,y) > 0$ in an empty market. □

We can rewrite the FOC as

$$k'(e(x,y)) = m(1, \theta(x,y))\,\eta \max[S(x,y), 0] \tag{16}$$

$$\gamma = \frac{m(1, \theta(x,y))}{\theta(x,y)}(1-\eta) \qquad if \quad S(x,y) > 0 \tag{17}$$

These two equations equate the marginal costs of search with the marginal benefits. Market tightness $\theta(x,y)$ is taken as given by each agent, but determined in equilibrium.

Finally, the assumption of a steady state means that the inflows to unemployment must equal the outflows:

$$s(l(x) - u(x)) = u(x) \int_0^1 e(x,y) m\left(1, \theta(x,y)\right) dy \tag{18}$$

2.1 Equilibrium

An equilibrium is defined by the surplus equation (6), the value of unemployment (1), the two marginal conditions for the value of search, (16) and (17), and the flow steady state condition. The full equilibrium is reproduced below:

$$S(x,y) = \frac{1}{r+s}\left(f(x,y) - rU(x,y)\right) \tag{19}$$

$$rU(x) = -\int k(e(x,y'))dy' + \int e(x,y') m\left(1, \theta(x,y')\right) \max\left[S(x,y'), 0\right] dy' \tag{20}$$

$$k'(e(x,y)) = m\left(1, \theta(x,y)\right) \eta \max\left[S(x,y), 0\right] \tag{21}$$

$$\gamma = \frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)}(1-\eta) S(x,y) \qquad if \quad S(x,y) > 0 \tag{22}$$

$$u(x) = \frac{sl(x)}{s + \int_0^1 e(x,y) m\left(1, \theta(x,y)\right) dy} \tag{23}$$

recalling that $\theta(x,y) = \frac{v(x,y)}{u(x)e(x,y)}$. The endogenous variables are $e(x,y)$, $v(x,y)$, $S(x,y)$, $u(x)$ and $U(x)$.

2.2 Sorting

In this section I trace the effects of supermodularity from the production function through the match surplus to search effort and matching rates. The main result relates complementarities in the production function (supermodularity) to the propensity of different workers to match with different firms (single crossing). This result can be understood as follows. First, consider the firm's problem. The firm posts vacancies in a market until the expected benefit equals the cost, γ. The benefit of posting vacancies has two parts, the arrival rate of workers $\frac{m(1,\theta)}{\theta}$ and the surplus $S(x,y)$. Fixing x, an increase in match surplus due to a change in y must bring an increase in market tightness θ. This is intuitive: there is free entry of vacancies, so higher surplus submarkets will have higher θ. Higher submarket tightness, in turn, incentivizes more search effort on the part of workers, since each unit of their search effort is more effective. Combining results, for a given x a higher surplus market has both higher tightness and more effort. This implies that higher surplus markets always have higher job finding rates. This is the critical result: given x, the transition rates to different job types is an increasing function of the surplus. In other words, the transition rates to different firms *share the same ordering* as the match surplus. Therefore if the surplus function is supermodular the matching rates are a monotone transform of a supermodular function. The ordinal implications of supermodularity are *quasi-supermodularity* and *single crossing*, discussed below.

First, some definitions:

Definition 1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if and only if

$$f(x,y') - f(x,y) < f(x',y') - f(x',y) \tag{24}$$

for all $x < x'$ and $y < y'$. If f is differentiable this implies $f_{12} > 0$ everywhere.

Supermodularity is usually identified with complementarities: increasing the first argument of f increases the marginal product of the second argument, and vice versa. The next definitions are special cases of those in Milgrom and Shannon (1994).

Definition 2. (Milgrom & Shannon) A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is satisfies the *single crossing property in* $(x;y)$ if, for $x < x'$ and $y < y'$,

$$f(x,y') > f(x,y) \quad \to \quad f(x',y') > f(x',y)$$

and

$$f(x,y') \geq f(x,y) \quad \rightarrow \quad f(x',y') \geq f(x',y)$$

This is called the single crossing property because it implies that $f(x,y')$, as a function of x, cuts $f(x,y)$ at most once and from below. Single crossing allows $f(x,y') = f(x,y')$ for an interval of x, but once $f(x,y') > f(x,y)$ it never returns. If $f(x,y)$ reflects something about the preferences of an agent typed x over outcomes y, there is another way to understand single crossing. If $y < y'$ and a low type x ranks y below y', then higher type x never reverse that ranking. In other words, if a low type x prefers high y to low y, then all higher ranked x must share that preference.

Definition 3. (Milgrom & Shannon) A continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is quasi-supermodular if it satisfies the single crossing property in both $(x;y)$ and $(y;x)$.

Quasi-supermodularity is the ordinal counterpart of supermodularity. Any increasing transformation of a supermodular function is quasi-supermodular.

Definition 4. A continuous function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is satisfies the *weak single crossing property in* $(x;y)$ if, for $x < x'$ and $y < y'$,

$$h(x,y') \geq h(x,y) \quad \rightarrow \quad h(x',y') \geq h(x',y) \tag{25}$$

Unlike single crossing, weak single crossing allows for $h(x',y') = h(x',y)$ when $h(x,y') > h(x,y)$. What weak single crossing forbids is $h(x',y') < h(x',y)$ when $h(x,y') > h(x,y)$. The main result will focus on weak single crossing.

With the definitions above we are ready to derive the main result. The main result depends on a sequence of relationships, starting with the match production function and ending with the matching rates. The following lemmas summarize these relationships, and Theorem 1 puts the pieces together.

It will be convenient to define $\tilde{S}(x,y)$ as the truncation of $S(x,y)$ at zero:

$$\tilde{S}(x,y) = \max\left[S(x,y), 0\right]$$

To begin, the relationship between $f(x,y)$ and $\tilde{S}(x,y)$ is fairly straightforward:

Lemma 2. *$S(x,y)$ is supermodular if and only if $f(x,y)$ is supermodular.*

This is a direct consequence of $S(x,y)$ being additive in $f(x,y)$ and $rU(x)$. Additivity means that $rU(x)$, a function of x only, cannot affect the cross partial. So $S_{12}(x,y)$ is proportional to $f_{12}(x,y)$, and in particular $S_{12}(x,y) \geq 0$ if and only if $f_{12}(x,y) \geq 0$.

Lemma 3. *$\tilde{S}(x,y)$ satisfies the weak single crossing property in $(x;y)$ and $(y;x)$ if $S(x,y)$ is supermodular.*

Proof. See appendix. □

Lemma 4. *$\theta(x,y)$ is strictly increasing in $\tilde{S}(x,y)$. $e(x,y)$ is increasing in $\tilde{S}(x,y)$. The worker transition rate $e(x,y)m\left(1,\theta(x,y)\right)$ is strictly increasing in $\tilde{S}(x,y)$.*

Proof. Recall the first order conditions:

$$k'(e(x,y)) = m\left(1,\theta(x,y)\right)\eta\tilde{S}(x,y) \tag{26}$$

$$\gamma = \frac{m\left(1,\theta(x,y)\right)}{\theta(x,y)}(1-\eta)S(x,y) \qquad if \quad S(x,y) > 0 \tag{27}$$

Take each part of the lemma separately:

$\theta(x,y)$ **is strictly increasing in** $\tilde{S}(x,y)$: Recall that $m(\cdot,\cdot)$ is a concave, constant returns function, with $m\left(\cdot,0\right) = 0$. Concavity implies $m\left(1,\theta\right)$ is a concave function of θ. Concavity and $m\left(1,0\right) = 0$ imply that $m_2\left(1,\theta\right) \leq m(1,\theta)\theta^{-1}$ everywhere. This inequality, in turn, means that $m(1,\theta)$ is decreasing in θ. Thus, considering equation (27), an increase in $\tilde{S}(x,y)$ must cause $\theta(x,y)$ to increase.

$e(x,y)$ **is strictly increasing in** $\tilde{S}(x,y)$: Given that $\theta(x,y)$ is strictly increasing in $\tilde{S}(x,y)$, from equation (26) an increase in $\tilde{S}(x,y)$ must result in an increase in $e(x,y)$.

$e(x,y)m\left(1,\theta(x,y)\right)$ **is strictly increasing in** $\tilde{S}(x,y)$: This follows trivially from the other two points.

□

Combining previous lemmas, I have the main result:

Theorem 1. *If $f(x,y)$ is supermodular then the worker transition rate $e(x,y)m(1,\theta(x,y))$ has the weak single crossing property in (x;y), i.e. the weak single crossing property in matching holds.*

Proof. Follows from Lemmas 2, 3, and 4, noting that the weak single crossing property is preserved under increasing transformations. $\square$

Thus, a worker find jobs more often with jobs that yield higher surplus, and less often with less desirable jobs. This is quite intuitive. The remarkable feature of this result is that it does not depend on the specifics of either the matching function or the costs of search, $k(\cdot)$.

Note that under absolute advantage (i.e. $f_1 > 0$, $f_2 > 0$), all worker expend more effort on high type vacancies than low type vacancies. This is a result of free entry for vacancies and directed vacancy search. In equilibrium, there is no opportunity cost for the firms (i.e., $rV(y) = 0$), so the surplus is increasing in firm type given any worker type. Thus all workers direct more effort to better firms, and high firms oblige by posting more vacancies than low type firms. It might be more plausible for worker search effort to be hump-shaped, with medium type worker directing most of their effort to medium type firms, with less effort going to very high or very low type firms. This can be shown to be the case in the multi-worker model of Section 4.7, where high type firms expand their employment, pushing down the marginal product of labor. In that model's equilibrium, low type workers will not expend much (or any) effort on high type vacancies. Another way to interpret this result is that while match production in Section 4.7, has absolute advantage, equilibrium marginal products have relative advantage.

3 Generalizations and Special Cases

In this section I cover several limiting cases of the model which are of special interest. The baseline model, slightly modified, can nest random worker search, directed search, as well as other cases.

3.1 Directed Search

In the standard directed search model, each worker and firm searches in a single submarket only. In the model of Section 2 workers spread their effort across many markets, because of decreasing returns to effort within each market. We can obtain approach the standard directed search model by making returns to effort nearly linear within markets, and adding a cost of effort summed across all markets.

Consider the following modification of (1):

$$rU(x) = -r\left(\int k(e(x,y'))dy'\right) + \int e(x,y')p(x,y')\left(W(x,y') - U(x,y')\right)dy' \tag{28}$$

where $r(\cdot)$ is a positive, increasing and convex function, capturing the increasing marginal disutility of effort *across all submarkets*. The first order condition for effort is then

$$r'\left(\int k(e(x,y'))dy'\right)k'(e(x,y)) = p(x,y)\left(W(x,y) - U(x,y)\right) \qquad if \quad e(x,y) > 0.$$

It can be confirmed that this modification leaves the main results of 2 unchanged. The only difference is that we have to condition some of the results in Lemma 4 on a fixed x. Lemma 5 states the new results:

Lemma 5. *Replace equation (1) with equation (28). Then $\theta(x,y)$ is strictly increasing in $\tilde{S}(x,y)$. $e(x,y)$ is increasing in $\tilde{S}(x,y)$* for a given x. *The worker transition rate $e(x,y)m\left(1,\theta(x,y)\right)$ is strictly increasing in $\tilde{S}(x,y)$* for a given x.

Proof. The proof is the same as Lemma 4, except we fix an x when evaluating the effects of $\tilde{S}(x,y)$ on $e(x,y)$. □

With the modified Lemma in hand, it is easy to show the main result holds in this model:

Theorem 2. *Replace equation (1) with equation (28). If $f(x,y)$ is supermodular then the worker transition rate $e(x,y)m\left(1,\theta(x,y)\right)$ has the weak single crossing property in (x;y).*

Proof. Follows from Lemmas 2, 3, and 5, noting that the weak single crossing property is preserved under increasing transformations. □

Now consider what happens as the function $k(\cdot)$ becomes linear: $k(e(x,y)) \to k^* \times e(x,y)$. Then the worker faces a linear tradeoff between different submarkets. The worker will generally opt for a corner solution, assigning positive effort only to the submarkets with maximal $m\left(1, \theta(x,y)\right)\eta\tilde{S}(x,y)$. On the firm side, as in section 2, a firm type y will only assign vacancies to submarkets with maximal $\frac{m(1,\theta(x,y))}{\theta(x,y)}(1-\eta)\tilde{S}(x,y)$. Thus search is fully directed.

Note that if firms have absolute advantage ($f_2(x,y) > 0$ for all x, y) then the directed search equilibrium is trivial: only the best firm type will enter, and they will employ everyone. Dropping the assumption of absolute advantage, as in Gautier and Teulings (2011), allows each worker type to have their own maximal firm type ("relative advantage"). In a fully directed search equilibrium, all the maximal firm types enter, each matching with their maximal worker types.

Assuming relative advantage can be problematic, since it means that firms cannot be ranked unambiguously. Each firm is more productive with some workers, and less productive with others. Then it is unclear what a "higher" firm type means. These problems are resolved in Section 4, with the introduction of multi-worker firms. With multi-worker firms, endogenous marginal products play the role that match production plays in single worker firms. With linear vacancy posting costs firm sizes adjust to endogenously create relative advantage among firm marginal products, even though firms can be unambiguously ranked in terms of productivity.

3.2 Random Worker Search

Another case of interest is when workers search randomly, visiting all submarkets with equal probability. Note that theorem 1 holds for any positive, increasing, convex effort function $k(e)$. Let

$$k(e) = e^{\alpha}.$$

Then as $\alpha \to \infty$ the marginal effort function, $k'(e)$ approaches the jump function

$$k'(e) = \begin{cases} 0 & if \;\; e < 1 \\ \infty & if \;\; e \geq 1 \end{cases}$$

As $\alpha \to \infty$, workers will tend to assign $e(x,y) \approx 1$ in all submarkets. It can be confirmed that in the limiting economy $e(x,y) = 1$ for all x, y. It is also easy to show that Theorem 1 holds in the limiting case. In particular, note that if $e(x,y) = 1$ everywhere, then the only relevant first order condition is the firm's:

$$\gamma = \frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)}(1-\eta)S(x,y) \qquad if \quad S(x,y) > 0$$

with $\theta(x,y) = \frac{v(x,y)}{u(x)}$. Following the same arguments as in section 2, it is clear that $\theta(x,y)$ is increasing in $\tilde{S}(x,y)$. When worker effort is degenerate the matching rate are just $m\left(1, \theta(x,y)\right)$, so matching rates are increasing in $\tilde{S}(x,y)$. So supermodularity implies matching rates have the weak single crossing property.

In this case workers search randomly, but vacancies are still fully directed. The next section shows that the results still hold when both workers and firms search randomly.

3.3 Random Search

So far I have focused on the case of free vacancy entry into all submarkets. In this section I consider a model where firms and workers search randomly, as in Shimer and Smith (2000). This kind of model has been extensively analyzed by Shimer and Smith (2000), Lopes de Melo (2013) and Law et al. (2014) among others, so the exposition here will be kept brief.

Let $a(y)$ be the fixed mass of type y firms in the economy. Each firm has a single job, which can be vacant or filled. Vacant jobs search for workers among the unemployed. Let $b(y)$ be the mass of vacant type y jobs. The mass of searching workers is given by $N = \int u(x)dx$, and the total mass of searching firms is $B = \int b(y)dy$. Search is random: both workers and firms search randomly. As is standard in such models, the flow of meetings is given by a constant returns, concave function $m(N, V)$. The rate at which an unemployed worker meets firms is given by

$$\frac{m(N,V)}{N} = m(1,\theta)$$

the rate at which a worker meets type y firms is

$$p(y) = \frac{v(y)}{V} m(1,\theta).$$

Without the effort margin, the unemployed worker's value function is

$$rU(x) = \eta \int p(y')\tilde{S}(x,y')dy'.$$

The value of a vacant job is symmetric:

$$rV(y) = (1-\eta) \int q(x')\tilde{S}(x',y)dx'$$

where

$$q(x') = \frac{u(x)}{N} m(\theta^{-1}, 1)$$

is the rate at which firms meet type x workers.

As Shimer and Smith show, the match surplus of this model can be written as

$$S(x,y) = \frac{f(x,y) - rU(x) - rV(y)}{r+s}$$

and matches form if and only if

$$S(x,y) \geq 0.$$

Note that the the rate at which type x workers find matches at type y jobs is $t(x,y) = p(y) \times \mathbf{1}\{S(x,y) \geq 0\}$. I will show that $t(x,y)$ has the weak single crossing property in matching whenever $f(x,y)$ is supermodular. The proof is by contradiction. Start by assuming that that $f(x,y)$ is supermodular and that the weak single crossing property fails. Then we can find $x < x'$ and $y < y'$ such that

$$t(x,y) < t(x,y')$$

and

$$t(x',y) > t(x',y')$$

The only way the second inequality can hold is if $S(x',y) \geq 0$ and $S(x',y') < 0$. If both surpluses were non-negative or both were negative, then the indicator values would coincide and we would have $t(x',y) = t(x',y') = p(y)$ or $t(x',y) = t(x',y') = 0$. Similarly, the first inequality implies that $S(x,y) < 0$ and $S(x,y') \geq 0$. Taking these inequality together I have

$$S(x,y) < S(x,y')$$

and

$$S(x', y) > S(x', y')$$

which violates supermodularity, since supermodularity would imply

$$S(x, y') - S(x, y) < S(x', y') - S(x', y).$$

Thus $S(x, y)$ cannot be supermodular. But note that $S(x, y)$ is supermodular if and only if $f(x, y)$ is supermodular. So I have a contradiction, and it must be that $t(x, y)$ has the weak single crossing property in matching whenever $f(x, y)$ is supermodular.

To borrow Shimer & Smith's notation, let $M(x) = \{y | t(x, y) > 0\}$ be the matching set for type x workers, and let $M(y)$ be defined symmetrically. What I have show is that under supermodularity, if $x \notin M(y)$ and $x \in M(y')$, then either $x' \in M(y)$ and $x' \in M(y')$ or $x' \notin M(y)$ and $x' \notin M(y')$. Put differently, if x shows a strict preference for y' over y, then x' cannot (strictly) reverse this preference. Note that these conditions are different from Shimer & Smith's definition of positive assortative matching. They define positive assortative matching as a situation where $x \in M(y')$ and $x' \in M(y)$ implies both $x \in M(y)$ and $x' \in M(y')$. Clearly, positive assortative matching implies that weak single crossing property, so weak single crossing is a weaker condition. In particular, when weak single crossing holds matching sets may be non-convex, which is ruled out by positive assortative matching.

4 Multi-worker Firms

So far I have assumed single worker firms (or equivalently, multi-worker firms with linear production and constant vacancy posting costs). In this section I consider multi-worker firms with decreasing returns in their labor inputs. This is of interest because multi-worker firms account for the majority of employment and output in the economy. Outside of the search literature it is standard to model firms as entities that freely choose the quantities of their inputs, subject to equilibrium prices and decreasing returns in production. This methodology is becoming increasing common in the search literature too, see Eeckhout and Kircher (2012), Schaal (2012), Kaas and Kircher (2015) and Elsby and Michaels (2008).

In this section I will show that the connection between supermodularity and weak single crossing survives in models with multi-worker firms. In the models of Section 2 the marginal product of a worker at a type y firm is always $f(x,y)$, regardless of equilibrium allocation of workers. The main complication of multi-worker firms is that the marginal product of a worker at a given firm is endogenous. Decreasing returns production implies that the marginal product of a worker depends not only on worker and firm type, but also the equilibrium level of employment at the firm. Match surplus, which drives sorting patterns, is the marginal product net of the value of unemployment. The key insight is that the endogenous firm size effects scales worker marginal products up and down together, at a given firm. This means that a given firm type will retain the same preference ordering over worker types, no matter what size the firm is in equilibrium. This result is based on a model with additive employment costs. A similar result obtains in a model where returns to scale are diminishing because of monopolistic competition.

Throughout, I focus on models where workers are perfect substitutes in production, but the tradeoff between worker types varies by firm. To be concrete, let $f(x,y)$ be match output, and let $n(x,y)$ be the mass of type x workers employed by a type y firm. Then the effective amount of labor supplied is

$$\int f(x,y)n(x,y)dx$$

For most of this section I focus on a production function of the form

$$F(n(\cdot,y)) = \int f(x,y)n(x,y)dx - c\left(\int\int_0^1 n(x,y)dx\right)$$

where $c()$ is a convex employment cost. The function $c(\cdot)$ restrains firm growth: if a firm hires too many workers the penalty $c(\cdot)$ will grow until it makes the marginal product zero. Firms face a linear tradeoff between worker types, and so each firm will generally only employ one type of worker. In Section 4.7 I analyze the case where firm size is constrained by monopolistic competition, or equivilently, decreasing returns in the labor aggregate, rather than decreasing returns in the number of workers.

In this section, I retain the general structure of the previous sections, with some important differences. Time is continuous and goes on forever. I focus exclusively on steady states. As in section 3.3, the number of firms of type y is fixed at $a(y)$. But now each firm can employ a continuum of workers. Let $n(x, y) : [0, 1] \to \mathbb{R}^+$ be the mass of type x workers employed at a type y firm, with $n(\cdot, y)$ representing the entire distribution of employment. Flow output of the firm is given by the positive, concave function $F(n(\cdot, y))$, which will be specified in more detail later. As before, firms post vacancies at a constant cost in (x, y)-specific submarkets. Each firm can post vacancies in as many (x, y) submarkets as they wish. Wages are continuously Nash bargained with each individual worker. Let $w(x, y)$ be the wage paid by a type y firm to a type x worker. This will generally depend on the employment distribution $n(\cdot, y)$ of the firm.

4.1 Firm Problem

The firm's problem can be written as

$$\Pi(n(\cdot, y)) = \max_{v(\cdot,y),n^+(\cdot,y)} \left(\frac{1}{1 + r\ dt}\right) \left\{ \left[F(n(\cdot, y)) - \int_0^1 w(x, y)n(x, y)dx - \gamma \int_0^1 v(x, y)dx\right] dt + \Pi\left(n^+(\cdot, y)\right) \right\} \tag{29}$$

subject to

$$n^+(x, y) \leq n(x, y)\,(1 - s\ dt) + q(x, y)v(x, y)\ dt \qquad \forall x \tag{30}$$

where $\Pi(n(\cdot, y, \cdot))$ is the present discounted value of profits conditional on the employment state variabwles $n(\cdot, y)$ at time t. The firm chooses the stock of vacancies in each submarket, $v(x, y)$, and employment at time $t + dt$, represented by $n^+(\cdot, y)$. Flow output is $F(n(\cdot, y))$. The flow costs are total wages paid its workers and total vacancy posting expenditures. The value of the firm at time $t + dt$ is

$\Pi\left(n^{+}(\cdot,y,\cdot)\right)$. The constraint (30) says that future employment of each worker type can be no greater than current employment less exogenous separations at rate s, plus the inflow of new candidates due to vacancy posting. The assumption that the economy (and the firm) is in steady state requires that

$$n^{+}(x,y)=n(x,y).$$

The main complication of multi-worker firms is that the marginal product, and thus the bargaining position, or each worker depends on every other worker at the firm. For example, if bargaining between the firm and a given worker breaks down and the worker separates, the marginal product of all remaining workers is higher (because $F(\cdot)$ is concave). The remaining workers will then extract a higher wage than they would have if the original worker had remained. Stole and Zwiebel (1996) solved this bargaining problem, though as Bruegemann et al. (2016) point out, the game that Stole and Zwiebel (1996) posit is not the one that leads to the proposed bargaining solution. The *match surplus* for each worker-firm pair depends not only on marginal physical output and the outside options, but also the effect of the worker on their co-worker's wages. Let $S^F(x,y)$ be the firm's surplus from employing a worker. Then I can write

$$rS^F(x,y)=\left[\frac{dF(n(\cdot,y))}{dn(x,y)}-w(x,y)-\int_0^1\frac{dw(x',y)}{dn(x,y)}n(x',y)dx\right]-sS^F(x,y). \tag{31}$$

See the appendix for derivations. This similar to the firm's surplus in equation (5). The match output, $f(x,y)$, is replaced with marginal production $\frac{dF(n(\cdot,y))}{dn(x,y)}$. The wage $w(x,y)$ plays the same role in both expression. The significant difference is that (31) includes the affect of the worker on co-worker wages. The marginal vacancy posting condition for (29) is then

$$\gamma=S^F(x,y)q(x,y)\qquad if\quad v(x,y)>0 \tag{32}$$

Again, see the appendix for details.

4.1.1 Production function

The specific production function used here is

$$F(n(\cdot, y)) = \int_0^1 f(x,y)n(x,y)dx - c\left(\int_0^1 n(x,y)dx\right) \tag{33}$$

where c is strictly positive, increasing and convex, and $f(x,y)$ is the *match output function*. Total production at the firm is the sum of match outputs less an employment cost term $c(\cdot)$. The employment cost reflects the increasing marginal costs of maintaining a large workforce, and pins down the size of the firm. We will keep the derivations general until Section 4.3, where substituting in the production function will be useful.

4.2 Worker Problem

The worker problem is largely unchanged from Section 2. Workers search in submarkets, subject to convex costs, and Nash bargain with their employers. As result, the same equations from Section 2 apply here. The value of an unemployed worker is

$$rU(x) = -\int k(e(x,y'))dy' + \int e(x,y')p(x,y')\left(W(x,y') - U(x,y')\right)dy', \tag{34}$$

the value of an employed workers is

$$rW(x,y) = w(x,y) - s\left(W(x,y) - U(x,y)\right), \tag{35}$$

and the first order condition for search effort is

$$k'(e(x,y)) = p(x,y)\left(W(x,y) - U(x,y)\right). \tag{36}$$

Combining (34) and (35), I obtain the worker's surplus as

$$rS^W(x,y) = w(x,y) - rU(x) - sS^W(x,y). \tag{37}$$

In Section 2 I could combine the worker and firm surpluses to cancel the wage $w(x,y)$ and obtain an expression for total match surplus. Following that logic here would result in an expression that still included the cross-wage effects $\int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx$. Thus, I must explicitly calculate the wage bargain to obtain a clean expression for match surplus.

4.3 Total surplus and Nash bargaining

With the worker and firm surpluses in hand, we can derive total match surplus. Sum equations (31) and (35) to obtain the total surplus as

$$rS(x,y) \;=\; \frac{dF(n(\cdot,y))}{dn(x,y)} - \int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx' - rU(x) - sS(x,y) \tag{38}$$

The worker's own wage, $w(x,y)$, is a transfer and so does not affect total surplus. However, the wages of the other workers at the firm *do* enter the surplus calculation. This is in contrast to Section 2, where adding up worker and firm surplus yielded total surplus as a function of output and the value of unemployment only.

Eliminating $\frac{dw(x',y)}{dn(x,y)}$ requires solving the wage bargaining problem. Generalized Nash bargaining implied that the worker and firm surplus are shared as

$$(1-\eta)S^W(x,y) = \eta S^F(x,y) \tag{39}$$

where η is the worker's bargaining power. We can substitute in the production function (33) and use equations (31), (35) and (39) to obtain the wage expression

$$w(x,y) \;=\; \eta\left(f(x,y) - c'(n(y)) - \int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx'\right) + (1-\eta)rU(x) \tag{40}$$

where $n(y) = \int_0^1 n(x,y)dx$ is total firm employment. See the appendix for derivations.

The right hand side of equation (40) has the standard form. The first term in parenthesis can be interpreted as the marginal product of the match: net physical match output $f(x,y) - c'(n(y))$ less the effect of the worker on coworker wages. The worker receives a share η of this. The second term is worker's outside option of unemployment.

Equation (40) is difficult to analyze because it defines a worker's wage in terms of the derivatives of the wages of all other workers at the firm. Cahuc et al. (2008) show that differential system defined by (40) has a relatively simple solution, given below:

$$w(x,y) \;=\; \int_0^1 z^{\frac{1-\eta}{\eta}} \left[f(x,y) - c'\left(z\int_0^1 n(x',y)dx'\right)\right] dz + (1-\eta)rU(x) \tag{41}$$

The solution eliminates all the wage derivatives, recasting the wage in terms of the derivatives of the production function. Substituting (41) into equation (37) and using $S^W(x,y) = \eta S(x,y)$ yields

$$S(x,y) \;=\; \frac{1}{r+s}\left\{\frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}} \left[f(x,y) - c'\left(z\int_0^1 n(x',y)dx'\right)\right] dz - rU(x)\right\} \tag{42}$$

This gives the match surplus as a function of gross match output $f(x,y)$, employment at the firm, and the value of unemployment. The important thing here is that $\int_0^1 n(x',y)dx' = n(y)$ is independent of x. Thus, the cross partial of $S(x,y)$ depends only on the cross partial of $f(x,y)$, as in Section 2:

$$S_{12}(x,y) = \frac{1}{r+s}\frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}} dz \times f_{12}(x,y)$$

This will imply that supermodularity still carries through to single crossing in matching.

4.4 Submarket matching

As in Section 2, matching each submarket is governed by the matching function $m(\cdot,\cdot)$. The only difference is that now the number of vacancies in a submarket is given by $a(y)v(x,y)$, rather than just $v(x,y)$. Let the job finding be

$$p(x,y) = m\left(1, \theta(x,y)\right) \tag{43}$$

where

$$\theta(x,y) = \frac{a(y)v(x,y)}{u(x)e(x,y)}$$

is submarket tightness. Symmetrically, the worker finding rate is

$$q(x,y) = \frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)}$$

Using $\eta S(x,y) = W(x,y) - U(x,y)$ and equation (43) in (34) I have the value of unemployment as

$$rU(x) = -\int k(e(x,y'))dy' + \eta \int e(x,y')m\left(1, \theta(x,y)\right) S(x,y')dy' \tag{44}$$

Making similar substitutions in (36) and (32) I obtain the worker and firm first order conditions as

$$k'(e(x,y)) = \eta m\left(1, \theta(x,y)\right) S(x,y) \tag{45}$$

$$\gamma = (1-\eta)\frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)} S(x,y). \tag{46}$$

Using the firm's employment constraint (30) I have

$$n(x,y) = n(x,y)(1-s) + v(x,y)\frac{m\left(1, \theta(x,y)\right)}{\theta(x,y)}$$

and steady state unemployment flows require

$$(l(x) - u(x))\, s = u(x) \int_0^1 e(x,y) m\,(1, \theta(x,y))\, dy$$

and finally

$$l(x) = u(x) + \int_0^1 a(y) n(x,y) dy$$

4.5 Equilibrium

Equilibrium is defined by the surplus equation (42), the value of unemployment (44), the worker first order condition (45), the firm first order condition (46), the firm's steady state employment constraint (30), and the steady state condition for unemployment flows. These are collected below:

$$S(x,y) = \frac{1}{r+s}\left\{\frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}}\left[f(x,y) - c'\left(z\int_0^1 n(x',y)dx'\right)\right]dz - rU(x)\right\} \tag{47}$$

$$rU(x) = -\int k(e(x,y'))dy' + \eta\int e(x,y')m\,(1,\theta(x,y))\, S(x,y')dy' \tag{48}$$

$$k'(e(x,y)) = \eta m\,(1,\theta(x,y))\, S(x,y) \tag{49}$$

$$\gamma = (1-\eta)\frac{m\,(1,\theta(x,y))}{\theta(x,y)} S(x,y) \tag{50}$$

$$n(x,y) = n(x,y)(1-s) + v(x,y)\frac{m\,(1,\theta(x,y))}{\theta(x,y)} \tag{51}$$

$$(l(x) - u(x))\, s = u(x)\int_0^1 e(x,y)m\,(1,\theta(x,y))\, dy \tag{52}$$

recalling that $\theta(x,y) = \frac{a(y)v(x,y)}{u(x)e(x,y)}$. Note that (51) and (52) together imply the population identity $l(x) = u(x) + \int_0^1 a(y)n(x,y)dy$, so it need not be included in the definition of equilibrium. The endogenous variables are $e(x,y)$, $u(x)$, $n(x,y)$, $v(x,y)$, $S(x,y)$, and $U(x)$.

4.6 Sorting

The sorting results from Section 2 go through nearly unchanged. The only distinction is in Lemma 2, showing that supermodular $f(x,y)$ implies supermodular $S(x,y)$. In Section 2 this amounted to noting that $f(x,y)$ and $rU(x)$ enter $S(x,y)$ additively, so the cross partial is a function of $f(x,y)$ only. Similar logic applies here. Because $c'\left(z\int_0^1 n(x',y)dx'\right)$ is independent of x, it drops out of the cross partial:

$$S_{12}(x,y) = \frac{1}{r+s}\frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}}dz \times f_{12}(x,y) \tag{53}$$

Thus supermodularity of $f(x,y)$ implies supermodularity of $S(x,y)$. Lemmas 3 and 4 and Theorem 1 go through unchanged. So the main result generalizes to multi-worker firm model:

Theorem 3. *In the multi-worker firm model, whenever $f(x,y)$ is supermodular then the worker transition rate $e(x,y)m(1,\theta(x,y))$ has the weak single crossing property in (x;y).*

Proof. Follows from equation (53) and Lemmas 3 and 4, noting that the weak single crossing property is preserved under increasing transformations. □

4.7 Endogenous output prices

So far I have assumed that firm output is sold for an exogenously fixed price. In this section, I analyze a model where firms face downward sloping demand for their products. Firms that grow large must accept a lower price. This setup is a generalization of the standard Dixit-Stiglitz model, where firms sell output to a consumer (or final goods producer) with a CES utility function (or a CES production function). I show that a modified version of the single crossing result survives in this environment. In particular, log-supermodular $f(x,y)$ implies weak single crossing. Log supermodularity means that $\log(f(x,y))$ is supermodular. It is a stronger condition than supermodularity under the assumption of absolute advantage.

In this section I replace the production function (33) with the function

$$F(n(\cdot,y)) = G\left(\int_0^1 f(x,y)n(x,y)dx\right) \tag{54}$$

where $G(\cdot)$ is assumed to be positive, increasing and concave. If we interpret $\int_0^1 f(x,y)n(x,y)dx$ as the physical output of the firm, then $G(\cdot)$ can be interpreted as the revenue function in a model of monopolistic competition. For example, say that the firm faces a demand curve of the form

$$p = Aq^{\rho-1},$$

where p is the price, q is the firm's quantity sold, $\rho \in (0,1)$, and A is a scale parameter. Then the revenue from selling $\int_0^1 f(x,y)n(x,y)dx$ units is

$$pq = A\left(\int_0^1 f(x,y)n(x,y)dx\right)^{\rho}.$$

In this example, $G(x) = Ax^{\rho}$.

An important distinction between (33) and (54) is that the latter imposes curvature on the production-weighted labor aggregate $\int_0^1 f(x,y)n(x,y)dx$, while the former imposes curvature on the unweighted labor aggregate $\int_0^1 n(x,y)dx$. This difference will drive the results in this section.

The change in production functions makes no difference to the general form of the worker and firm problems. The surplus equation (38) is written in terms of the general production function $F(n(\cdot,y))$, and so remains valid. However, the wage bargain needs to be rederived. Substituting the in the production function (54) and using equations (31), (35) and (39), I get the intermediate wage expression

$$w(x,y) = \eta\left(G'(e(y))f(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)}n(x',y)dx'\right) + (1-\eta)rU(x) \tag{55}$$

where $e(y) = \int_0^1 f(x,y)n(x,y)dx$ is total efficiency units of labor supplied to the firm. Equation (55) is similar to equation (40). The "direct" marginal product of the worker is $G'(e(y))f(x,y)$ - their direct impact on production/revenue. As in equation (40), the effects on co-worker wages is present. Luckily, this problem can also be solved with the tools in Cahuc et al. (2008). The system of equations defined by (55) has the solution

$$w(x,y) \quad = \quad f(x,y)\int_0^1 z^{\frac{1-\eta}{\eta}} G'\left(z\int_0^1 n(x',y)f(x',y)dx'\right)dz + (1-\eta)rU(x)$$

Plugging this into (37) and using $S^W(x,y) = \eta S(x,y)$, I have

$$S(x,y) = \frac{1}{r+s}\left\{f(x,y)\frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}} G'\left(z\int_0^1 n(x',y)f(x',y)dx'\right)dz - rU(x)\right\}.$$

Letting

$$P(y) = \frac{1}{\eta}\int_0^1 z^{\frac{1-\eta}{\eta}} G'\left(z\int_0^1 n(x',y)f(x',y)dx'\right)dz$$

I can write the surplus as

$$S(x,y) = \frac{1}{r+s}\left\{f(x,y)P(y) - rU(x)\right\} \tag{56}$$

This expression has a significant difference from its counterpart (42). The critical feature of (42) is that the firm-specific term, $\int_0^1 z^{\frac{1-\eta}{\eta}} c'\left(z\int_0^1 n(x',y)dx'\right)dz$ enters *additively*, and thus does not affect the cross partial. In contrast, the firm-specific term in (56) enters *multiplicatively*, so the cross partial is is a more complicated expression:

$$S_{12}(x,y) = \frac{1}{r+s}\left\{f_{12}(x,y)P(y) + f_1(x,y)P'(y)\right\}.$$

It is not clear what the sign of $P'(y)$ will be in equilibrium. It is a function of the second derivative of $G(\cdot)$ and the equilibrium size of each firm.

Although the strong connection between $f_{12}(x,y)$ and $S_{12}(x,y)$ is broken here, there is still a structure to the problem that can be exploited. I will show that log-supermodular $f(x,y)$ preserves the weak single crossing property in the present model.

Consider what (56) says about a worker's preference over two firm types, $y < y'$: Surplus is weakly greater at y' (i.e., $S(x,y) \leq S(x,y')$) if and only if

$$f(x,y)P(y) \leq f(x,y')P(y')$$

which is equivalently written as

$$\frac{P(y)}{P(y')} \leq \frac{f(x,y')}{f(x,y)}. \tag{57}$$

Similarly, another worker x', with $x < x'$, has weakly higher surplus at y' ($S(x'y) \leq S(x',y')$) if and only if

$$\frac{P(y)}{P(y')} \leq \frac{f(x',y')}{f(x',y)}. \tag{58}$$

I want to find conditions under which (57) implies (58). This implication will always hold if

$$\frac{f(x,y')}{f(x,y)} \leq \frac{f(x',y')}{f(x',y)} \tag{59}$$

for all $x < x'$ and $y < y'$. As it turns out, (59) is the definition of log-supermodularity. Thus, if $f(x,y)$ is log-supermodular then (57) always implies (58). This mean that under log-supermodular $f(x,y)$:

$$S(x,y) \leq S(x,y') \quad \rightarrow \quad S(x',y) \leq S(x',y')$$

whenever $x < x'$ and $y < y'$. That is, $S(x,y)$ has the weak single crossing property in $(x;y)$.

Once we have a single crossing property for $S(x,y)$, the other results in section (2) flow naturally. Thus, single crossing in worker transition rates can be preserved:

Theorem 4. *In the multi-worker firm model with endogenous prices, if $f(x,y)$ is log-supermodular then the worker transition rate $e(x,y)m\left(1,\theta(x,y)\right)$ has the weak single crossing property in (x;y).*

It is not too surprising that we need log-supermodularity to get weak single crossing. In fact, log-supermodularity is the sufficient condition for assortative matching a frictionless model with the production function (54), see Costinot (2009) and Costinot and Vogel (2010). Thus it makes sense that under generalized search frictions same types of production functions would generate single crossing.

In this model, firms with extremely high match productivity $f(x, y)$ will naturally expand their workforce until the marginal product of labor falls toward an equilibrium value. Note that the term $P(y)$ is a function of the marginal (and infra-marginal) product of labor at the firm. As the firm expands, $P(y)$ falls, scaling down the surplus of all matches. In equilibrium, a high type firm may have a very low value of $P(y)$, and thus may be an unprofitable match for a low type worker. Thus, in equilibrium workers will not all agree on their rankings of firms. Under supermodularity, good workers will be better off with good firms, but low type workers will be better off matches with low type firms. The endogenous firm size choice plays the same role that vacancy opportunity cost plays in Shimer and Smith (2000), incentivizing sorting and inducing hump-shaped preferences. Another way to interpret this result is that while match production may absolute advantage, equilibrium marginal products have relative advantage.

5 Conclusion

This paper has extended the study of sorting in a number of direction. I have laid out a general model of search, which can accomodate varying degrees random and directed search by workers. I have shown that complementarities in production imply a weak single crossing property in worker transition rates. The single crossing property is weaker than tranditional notions of assortative matching, but appears more robust. Supermodularity implies single crossing not only in the partially directed search models, but also in Shimer and Smith (2000).

The results also extend to multi-worker firms. I show that under a certain type of production function the same supermodularity-single crossing property holds. The final section studies a model of firms that is commonly used in the literature, a generalizations of the Dixit-Stiglitz model. In the frictionless version of that model, log-supermodularity is required for assortative matching. I show that under general search frictions log-supermodularity implies single crossing. Thus the results generalize in a natural way.

A Omitted Proofs and Derivations

Proof of Lemma 3. I will show that $\tilde{S}(x,y)$ satisfies the weak single crossing property in $(x;y)$. The proof for $(y;x)$ is symmetric. Assume that $S(x,y)$ is supermodular. If weak single crossing was violated then there would be $x < x'$ and $y < y'$ such that

$$\tilde{S}(x,y) < \tilde{S}(x,y')$$

and

$$\tilde{S}(x',y) > \tilde{S}(x',y')$$

But these inequalities imply

$$S(x,y) < S(x,y')$$

and

$$S(x',y) > S(x',y') \tag{60}$$

which contradicts the assumption of supermodularity, so I am done. □

Derivation of (31) and (32)

Start with the firm's problem (29) It is assumed that the firm can freely reduce employment at any time. $n(\cdot,y)$ is a state variable. Let $n^*(\cdot,y)$ be employment after any employment reductions but before production. The Lagrangian for the firm's problem is

$$\begin{aligned}
L(n(\cdot,y)) &= \left(\frac{1}{1+r\,dt}\right)\left\{\left[F(n^*(\cdot,y)) - \int_0^1 w(x',y)n^*(x',y)dx' - \gamma\int_0^1 v(x',y)dx'\right]dt + \Pi\left(n^+(\cdot,y)\right)\right\} \\
&+ \left(\frac{1}{1+r\,dt}\right)\int_0^1 \alpha(x')\left[\left(n^*(x',y)\left(1-s\,dt\right) + q(x',y)v(x',y)\,dt\right) - n^+(x',y)\right]dx' \\
&+ \left(\frac{1}{1+r\,dt}\right)\int_0^1 \mu(x')\left[n(x',y) - n^*(x',y)\right]dx'
\end{aligned}$$

with choice variables $n^*(\cdot,y), n^+(\cdot,y), v(\cdot,y)$. Note that $\alpha(x)$ is the multiplier on the constraint

$$n^+(x,y) \leq n^*(x,y)\left(1-s\,dt\right) + q(x,y)v(x,y)$$

and $\mu(x)$ is the multiplier on the constraint

$$n^*(x,y) \leq n(x,y)$$

The first order condition with respect to $v(x,y)$ is:

$$\left(\frac{1}{1+r\,dt}\right)\left\{-\gamma dt + \alpha(x)q(x)\,dt\right\} = 0$$

$$\gamma = \alpha(x)q(x) \tag{61}$$

And the first order condition with respect to $n^*(x,y)$ is:

$$\left(\frac{1}{1+r\,dt}\right)\left\{\left[\frac{dF(n^*(\cdot,y))}{dn^*(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn^*(x,y)}n^*(x',y)dx'\right]dt + \alpha(x)\left(1-s\,dt\right) - \mu(x)\right\} \leq 0 \tag{62}$$

with

$$\left(\frac{1}{1+r\,dt}\right)\left\{\left[\frac{dF(n^*(\cdot,y))}{dn^*(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn^*(x,y)}n^*(x',y)dx'\right]dt + \alpha(x)\left(1-s\,dt\right) - \mu(x)\right\} < 0 \rightarrow n^* = 0$$

Finally the first order condition with respect to $n^{+}(x,y)$ is

$$\frac{d\Pi\left(n^{+}(\cdot,y)\right)}{dn^{+}(x,y)} \leq \alpha(x) \tag{63}$$

with

$$\frac{d\Pi\left(n^{+}(\cdot,y)\right)}{dn^{+}(x,y)} < \alpha(x) \rightarrow n^{+}(x,y) = 0$$

And the envelope condition is

$$\frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)} = \left(\frac{1}{1+r\ dt}\right)\mu(x) \tag{64}$$

Assume for now that $n(x,y) > 0$. Then steady state implies that $n^{*}(x,y) > 0$, $n^{+}(x,y) > 0$ too. Then (62) and (63) hold with equality. Substitute (63) and (64) into (62), and impose $n^{*}(x,y) = n(x,y)$ to get

$$\left(\frac{1}{1+r\ dt}\right)\left\{\left[\frac{dF(n(\cdot,y))}{dn(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)}n(x',y)dx\right]dt + (1-s\ dt)\frac{d\Pi\left(n^{+}(\cdot,y)\right)}{dn^{+}(x,y)}\right\} - \frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)} = 0$$

Steady state implies $\frac{d\Pi\left(n^{+}(\cdot,y)\right)}{dn^{+}(x,y)} = \frac{d\Pi(n(\cdot,y))}{dn(x,y)}$, so we have

$$\frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)} = \left(\frac{1}{1+r\ dt}\right)\left\{\left[\frac{dF(n(\cdot,y))}{dn(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)}n(x',y)dx\right]dt + (1-s\ dt)\frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)}\right\}$$

$$r\frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)} = \left[\frac{dF(n(\cdot,y))}{dn(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)}n(x',y)dx\right] - s\frac{d\Pi\left(n(\cdot,y)\right)}{dn(x,y)} \tag{65}$$

for $n(x,y) > 0$.

Now define $S^F(x,y)$ as the function satisfying

$$rS^F(x,y) = \left[\frac{dF(n(\cdot,y))}{dn(x,y)} - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)}n(x',y)dx\right] - sS^F(x,y) \tag{66}$$

for any x, y. While $\frac{d\Pi(n(\cdot,y))}{dn(x,y)}$ is truncated at zero, $S^F(x,y)$ can go below zero if $n(x,y) = 0$. This completes the derivation of equation (31).

Returning to the FOC (61), I can substitute in $S^F(x,y) = \frac{d\Pi(n(\cdot,y))}{dn(x,y)} = \alpha(x)$ whenever $n(x,y) > 0$, so I have

$$\gamma = S^F(x,y)q(x,y) \qquad if \quad v(x,y) > 0 \tag{67}$$

which completes the derivation of (32).

Derivation of (40)

Substitute (33) into (31) to get:

$$rS^F(x,y) = f(x,y) - c'(n(y)) - w(x,y) - \int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx - sS^F(x,y). \tag{68}$$

Substitute this and 37 into 39 to get

$$(1-\eta)\left\{w(x,y) - rU(x) - sS^W(x,y)\right\} = \tag{69}$$

$$\eta\left\{f(x,y) - w(x,y,\varepsilon) - c'(n(y)) - \int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx'd - sS^F(x,y,\varepsilon)\right\} \tag{70}$$

and finally:

$$w(x,y) = \eta\left\{f(x,y) - c'(n(y)) - \int_0^1 \frac{dw(x',y)}{dn(x,y)} n(x',y)dx'\right\} + (1-\eta)rU(x) \tag{71}$$

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